The Covariant Picard Groupoid in Differential Geometry*

Stefan Waldmann**
Fakultät für Mathematik und Physik
Albert-Ludwigs-Universität Freiburg
Physikalisches Institut
Hermann Herder Straße 3
D 79104 Freiburg
Germany

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Abstract

In this article we discuss some general results on the covariant Picard groupoid in the context of differential geometry and interpret the problem of lifting Lie algebra actions to line bundles in the Picard groupoid approach.

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Contents

1 Introduction

In this work we would like to illustrate and exemplify some general results from [16] where the general framework of a Morita theory which is covariant under a given Hopf algebra was studied. One main motivation to do so is coming from (deformation) quantization theory [3], see e.g [12, 14] for recent reviews. Here Morita equivalence provides an important notion of equivalence of

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**E-mail: Stefan.Waldmann@physik.uni-freiburg.de

observable algebras [6, 31]. In particular, on cotangent bundles the condition for star products to be Morita equivalent is shown to coincide with Dirac's integrality condition for magnetic charges of a background magnetic field [6] leading to a natural interpretation of Morita equivalence also in more general situations.

From the differential geometric point of view, it is a natural question whether all these techniques as developed in [5, 7–9, 31] can be made compatible with a certain given symmetry of the underlying manifold. On the purely algebraic level, a fairly general notion of 'symmetry' is that of a Hopf algebra action of a given Hopf algebra. In [16] we studied this type of symmetry in the general situation.

From this general framework we shall specialize now into two directions: on one hand, the algebras on which the symmetry acts and whose Morita theory shall be studied will now be commutative: we are interested in the algebra of functions $C^\infty(M)$ on a manifold. On the other hand, the symmetry in question will either be coming from a Lie group action on M or from a Lie algebra action as its infinitesimal counterpart.

It is well-known that two commutative algebras are Morita equivalent if and only if they are isomorphic, see e.g. the textbook [19], whence for commutative algebras Morita equivalence seems to be a useless notion. However, this is not true as things become interesting if one asks in addition in how many ways two algebras can be Morita equivalent compared to the ways in which they can be isomorphic. It turns out that in general there are new possibilities which makes Morita theory interesting even in the commutative framework.

This phenomenon is precisely encoded in the so-called Picard groupoid which we shall compute for the case of function algebras. This way, we find an interesting and non-trivial class of examples illustrating the general ideas of [16]. Moreover, it will also be of independent interest as we are now able to re-interpret several well-known problems and results in differential geometry from a Morita theoretic point of view. Finally, the commutative situation with the algebras being function algebras $C^\infty(M)$ is expected to be the starting point for a discussion of Morita equivalence of star products as in [6] but now being compatible with a symmetry of the classical phase space [15].

The article is organized as follows: In Section 2 we recall some well-known arguments why one should and how one can pass from a geometric to a more algebraic description of differential geometry. The next section is devoted to a general discussion on Morita theory in different flavours, taking into account specific structures of the algebras in question. Here we are mainly interested in *-involutions and notions of positivity. In Section 4 we add one more structure to be preserved by Morita theory, namely a symmetry which we model by a Hopf algebra action. This can be specialized to group actions and Lie algebra actions. The last section contains some new material, namely the explicit computation of a certain part of the Lie algebra covariant Picard group.

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2 From Geometry to Algebra

In some sense, differential geometric methods in mathematical physics correspond mainly to classical theories: Hamiltonian mechanics on a symplectic or Poisson manifold M is one prominent example. On the other hand, quantum theories require a more algebraic approach: here the uncertainty relations in physics are modelled mathematically by non-trivial commutation relations

between observables in some noncommutative algebra, the observable algebra. Thus *quantization* in a very broad sense can be understood as the passage from geometric to noncommutative algebraic structures. An intermediate step is of course to encode the geometric structures on M in algebraic terms based on the commutative algebra of functions $C^\infty(M)$ where, in view of applications to quantization, it is convenient to consider complex-valued functions.

Then it is a folklore statement (Milnor's exercise) that one can recover the smooth manifold M from the *-algebra $C^\infty(M)$. More specifically: every *-homomorphism $\Phi : C^\infty(M) \longrightarrow C^\infty(N)$ between function algebras is actually of the form $\Phi = \phi^*$ with some smooth map $\phi : N \longrightarrow M$ between the underlying manifolds, see e.g. [13, 24] for a recent discussion. Thus the category of *-algebras with *-homomorphisms as morphisms becomes relevant to differential geometry.

The 'dictionary' to translate geometric to algebraic terms, which is also one of the cornerstones of Connes' noncommutative geometry [11], can be extended in various directions. We mention just one further example also relevant to Morita theory: by the well-known Serre-Swan theorem, see e.g. [29], the (complex) vector bundles $E \longrightarrow M$ correspond to finitely generated projective modules over the algebra $C^\infty(M)$ via $E \leftrightarrow \Gamma^\infty(E)$. In more geometric terms this means that for any vector bundle there exists another vector bundle $F \longrightarrow M$ such that $E \oplus F$ is a trivial vector bundle. This is of course the key to relate the algebraic K_0-theory of $C^\infty(M)$ to the topological K^0-theory of M.

Let us now turn to Morita theory. Its first motivation came from the question what one can say about two algebras $\mathcal{A}$ and $\mathcal{B}$ provided one knows that their categories of (left) modules are equivalent, see [19,23]. Clearly, in view of applications to quantum mechanics a good understanding of the more specific modules given by *-representations of the observable algebras on (pre) Hilbert spaces is crucial for any physical interpretation. As we shall not start to define what a reasonable category of modules over the *-algebra $C^\infty(M)$ should be —though this can perfectly be done, see e.g. [7, 27, 31] and references therein— we take a different motivation which will lead essentially to the same structures. The idea is to take the category of *-algebras, keep the objects and enhance the notion of morphisms. This can be expected to be interesting as for function algebras we already know what the 'ordinary' morphisms are: pull-backs by smooth maps. Thus a generalization would lead to a generalization of smooth maps between manifolds, when we translate things back using our 'dictionary'. In particular, it might happen that algebras become isomorphic in this new, enhanced category (which will turn out to be not the case for function algebras) and one might have more 'automorphisms' of a given algebra (which will indeed be the case for function algebras). In general, the invertible morphisms in a category form a (large) groupoid in the obvious sense which is called the Picard groupoid of the category. Thus a major step in understanding the whole category is to consider its Picard groupoid of invertible arrows first.

In principle, the whole idea should be familiar from geometric mechanics as one example of enhancing a category by allowing more general morphisms is given by the symplectic 'category': first one considers symplectic manifolds as objects and symplectomorphisms as morphisms. Though this is a reasonable choice to look at, it turns out to be rather boring as the choice for the morphisms is too restrictive. More interesting is the 'category' where one considers morphisms to be canonical relations, see e.g. [2]. However, this is no longer an honest category since the composition of morphisms is only defined when certain technical requirements like clean intesections of the canonical relations are fulfilled. Nevertheless this 'symplectic category' is by far more interesting now.

Other examples are the Morita theory for (integrable) Poisson manifolds by Xu [32] as well as the Morita theory of Lie groupoids, see e.g. [22].

3 Morita Equivalence in Different Flavours

After having outlined the general ideas in the previous section we should start being more concrete now. As warming up we discuss the 'enhancing of the category' for the category of unital algebras with usual algebra morphisms first, see e.g. [4, 19] for this classical approach.

Here the generalized morphisms are the *bimodules*: For two algebras $\mathcal{A}$ and $\mathcal{B}$ a $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{E}$, which we shall frequently denote by ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ to indicate that $\mathcal{B}$ acts from the left while $\mathcal{A}$ acts from the right, is considered as an arrow $\mathcal{A} \longrightarrow \mathcal{B}$.

Why does this give a reasonable notion of morphisms? In particular, we have to define the composition of morphisms. Thus let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ be bimodules then their tensor product ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ over $\mathcal{B}$ is a $(\mathcal{C}, \mathcal{A})$-bimodule and hence an arrow $\mathcal{A} \longrightarrow \mathcal{C}$. However, this is not yet an associative composition law as for three bimodules ${}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}}$, ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$, ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ we have a *canonical isomorphism*

$$
{}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}} \otimes_{\mathcal{C}} \left({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \right) \cong \left({}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}} \otimes_{\mathcal{C}} {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \right) \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \tag{3.1}
$$

as $(\mathcal{D}, \mathcal{A})$-bimodules but *not equality*. The way out is to use isomorphism classes of bimodules as arrows instead of bimodules themselves. Then the tensor product becomes indeed associative and the isomorphism class of the canonical bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ serves as the identity morphism of the object $\mathcal{A}$ since we use unital algebras for simplicity.

The final restriction we have to impose is that in a category the morphism space between two objects has to be a set, which is a priori not clear in our enhanced category. Therefor one should pose additional constraints on the bimodules like finitely generatedness. However, we shall ignore these subtleties in the following as the new notion of isomorphisms in this category will be unaffected anyway.

However, we still have to show that we really get an extension of our previous notion of morphisms. Thus let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an algebra homomorphism. Then on $\mathcal{B}$ we define a right $\mathcal{A}$-module structure by $b \cdot_{\Phi} a = b\Phi(a)$ and obtain a bimodule ${}_{\mathcal{B}}\mathcal{B}^{\Phi}_{\mathcal{A}}$. Its isomorphism class is denoted by $\ell(\Phi)$. It is easy to see that $\ell(\Phi \circ \Psi) = \ell(\Phi) \circ \ell(\Psi)$ and $\ell(\mathsf{id}_{\mathcal{A}})$ is the class of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ whence our previous notion of morphisms is indeed contained in the new one.

If we denote this new category by ALG then two unital algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent* iff they are isomorphic in ALG. Without going into the details this is equivalent to the existence of a certain bimodule which is 'invertible' with respect to the composition $\otimes$. In fact, such bimodules can be characterized rather explicitly, see e.g. [19].

The isomorphism classes of these invertible bimodules constitute now the *Picard groupoid* of this category ALG which we shall denote by Pic. The invertible arrows from $\mathcal{A}$ to $\mathcal{B}$ are denoted by $\mathsf{Pic}(\mathcal{B}, \mathcal{A})$ while the isotropy group of this groupoid at the local unit $\mathcal{A}$ is denoted by $\mathsf{Pic}(\mathcal{A})$, the *Picard group* of $\mathcal{A}$.

The map ℓ induces now a group homomorphism such that

$$
1 \longrightarrow \mathsf{InnAut}(\mathcal{A}) \longrightarrow \mathsf{Aut}(\mathcal{A}) \stackrel{\ell}{\longrightarrow} \mathsf{Pic}(\mathcal{A}) \tag{3.2}
$$

is exact, whence in the commutative case, the automorphism group of $\mathcal{A}$ is a subgroup of the Picard group $\mathsf{Pic}(\mathcal{A})$. Finally, it can be shown that for commutative $\mathcal{A}$, the exact sequence (3.2) is split whence

$$
\mathsf{Pic}(\mathcal{A}) = \mathsf{Aut}(\mathcal{A}) \ltimes \mathsf{Pic}_{\mathcal{A}}(\mathcal{A}), \tag{3.3}
$$

where the subgroup $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ consists of the symmetric invertible bimodules, i.e. those where $a \cdot x = x \cdot a$ for all $x \in \mathcal{E}$ and $a \in \mathcal{A}$. Then $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ is called the *commutative* or *static Picard group*, see e.g. [8, 10] for a discussion and further references.

It is a well-known theorem in Morita theory that for unital algebras $\mathcal{A}$, $\mathcal{B}$ the equivalence bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ are certain finitely generated projective right $\mathcal{A}$-modules such that $\mathsf{Hom}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}) \cong \mathcal{B}$. Coming back to our example $\mathcal{A} = C^\infty(M)$ we see, using the Serre-Swan theorem, that the only candidates for the symmetric self-equivalence bimodules are the sections $\Gamma^\infty(L)$ of a complex *line bundle*. In fact, it turns out that $\Gamma^\infty(L)$ is indeed invertible with inverse given by the class of $\Gamma^\infty(L^*)$ since $\Gamma^\infty(L) \otimes_{C^\infty(M)} \Gamma^\infty(L) \cong \Gamma^\infty(L^* \otimes L) \cong C^\infty(M)$ as $C^\infty(M)$-bimodules. This shows that the static Picard group of $C^\infty(M)$ is just the 'geometric' Picard group, i.e. the group of isomorphism classes of complex line bundles with the tensor product as multiplication. Using the Chern class to classify complex line bundles then gives according to (3.3)

$$\mathsf{Pic}(C^\infty(M)) = \mathrm{Diffeo}(M) \ltimes \check{\mathrm{H}}^2(M, \mathbb{Z}), \tag{3.4}$$

where the semidirect product structure comes from the usual action of diffeomorphisms on $\check{\mathrm{H}}^2(M, \mathbb{Z})$. In general, all Morita equivalence bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ for $\mathcal{A} = C^\infty(M)$ are isomorphic to some $\Gamma^\infty(E)$ with a vector bundle $E \longrightarrow M$ of non-zero fibre dimension. Moreover, $\mathcal{B}$ has to be isomorphic to $\Gamma^\infty(\mathsf{End}(E))$. Thus for function algebras $C^\infty(M)$ we have a complete description of the Picard groupoid.

We shall now specialize our notion of Morita equivalence: we have already argued that the *-involution of $C^\infty(M)$ should be taken into account when having applications to quantization in mind. Moreover, one can include notions of positivity into Morita theory. One defines a linear functional $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ to be *positive* if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. Then an element $a \in \mathcal{A}$ is called *positive* if $\omega(a) \geq 0$ for all positive linear functionals ω of $\mathcal{A}$, see [7, 27, 30, 31] for a detailed discussion. It is clear that for applications to quantum theories such notions of positive functionals are crucial as they encode expectation value functionals and hence the physical *states* for the observable algebra.

In particular, for $\mathcal{A} = C^\infty(M)$ one finds that positive linear functionals are precisely the integrations with respect to compactly supported positive Borel measures. This follows essentially from Riesz' representation theorem, see [5, App. B]. From this it immediately follows that $f \in C^\infty(M)$ is positive iff $f(x) \geq 0$ for all $x \in M$, whence the above, purely algebraic definition reproduces the usual notion.

We can now state the definition of *-Morita equivalence [1] and strong Morita equivalence bimodules, see [26] as well as [20] for Rieffel's original formulation in the context of C^*-algebras and [5, 7] for the general case of *-algebras. Instead of describing the 'enhanced category' way, we directly give the definition in terms of bimodules which is entirely equivalent, see [7].

Definition 3.1 *A *-Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule together with inner products*

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A} \tag{3.5}$$

and

$${}_{\mathcal{B}}\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{B} \tag{3.6}$$

such that for all $x, y, z \in \mathcal{E}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have:

1. *$\langle \cdot, \cdot \rangle_{\mathcal{A}}$ (resp. ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$) is linear in the right (resp. left) argument.*

2. *$\langle x, y \cdot a \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}}\, a$ and ${}_{\mathcal{B}}\langle b \cdot x, y \rangle = b\, {}_{\mathcal{B}}\langle x, y \rangle$.*

3. *$\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ and ${}_{\mathcal{B}}\langle x, y \rangle = {}_{\mathcal{B}}\langle y, x \rangle^*$.*

4. *$\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ are non-degenerate.*

5. $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ *and* ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ *are full.*

6. $\langle x, b \cdot y \rangle_{\mathcal{A}} = \langle b^* \cdot x, y \rangle_{\mathcal{A}}$ *and* ${}_{\mathcal{B}}\langle x, y \cdot a \rangle = {}_{\mathcal{B}}\langle x \cdot a^*, y \rangle$.

7. ${}_{\mathcal{B}}\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}$.

If in addition the inner products are completely positive then ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *is called a strong Morita equivalence bimodule.*

Here $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is called *full* if the $\mathbb{C}$-span of all elements $\langle x, y \rangle_{\mathcal{A}}$ is the whole algebra $\mathcal{A}$; in general these elements constitute a *-ideal. Moreover, $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is called *completely positive* if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in \mathcal{E}$ the matrix $(\langle x_i, x_j \rangle_{\mathcal{A}}) \in M_n(\mathcal{A})$ is positive in the *-algebra $M_n(\mathcal{A})$.

The composition of bimodules is again the tensor product where on ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ the $\mathcal{A}$-valued inner product is now defined by Rieffel's formula

$$\langle x \otimes \phi, y \otimes \psi \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} = \left\langle \phi, \langle x, y \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot \psi \right\rangle_{\mathcal{A}}^{\mathcal{E}}, \tag{3.7}$$

and analogously for the $\mathcal{C}$-valued inner product. It is then a non-trivial theorem that this is indeed completely positive again, if the inner products on $\mathcal{E}$ and $\mathcal{F}$ have been completely positive [7]. Passing to isometric isomorphism classes one can show that this gives a groupoid: the *-Picard groupoid Pic^* and the strong Picard groupoid $\mathsf{Pic}^{\mathrm{str}}$, respectively. In particular, the local unit at $\mathcal{A}$ is given by the isometric isomorphism class of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ equipped with the inner products

$$\langle a, b \rangle_{\mathcal{A}} = a^* b \quad \text{and} \quad {}_{\mathcal{A}}\langle a, b \rangle = a b^*. \tag{3.8}$$

More generally, $\mathcal{A}^n$, viewed as $(M_n(\mathcal{A}), \mathcal{A})$-bimodule equipped with the canonical inner products

$${}_{M_n(\mathcal{A})}\langle x, y \rangle = \sum_{i=1}^{n} x \cdot \langle y, \cdot \rangle_{\mathcal{A}} \quad \text{and} \quad \langle x, y \rangle_{\mathcal{A}} = \sum_{i=1}^{n} x_i^* y_i \tag{3.9}$$

implements the strong Morita equivalence between $\mathcal{A}$ and $M_n(\mathcal{A})$.

Since we simply can forget the additional structures we obtain canonical groupoid morphisms

$$\begin{tikzcd} \mathsf{Pic}^{\mathrm{str}} \arrow[rr] \arrow[dr] & & \mathsf{Pic}^* \arrow[dl] \\ & \mathsf{Pic} & \end{tikzcd}, \tag{3.10}$$

which have been studied in [7]: in general, none of them is surjective nor injective, even on the level of the Picard groups.

The geometric interpretation of the inner products is that they correspond to *Hermitian fiber metrics* on the corresponding line bundles or vector bundles, respectively: Indeed, this can be seen easily from the very definitions. Since up to isometry there is only one positive Hermitian fiber metric on a given line bundle we have in the case of $\mathcal{A} = C^\infty(M)$

$$\mathsf{Pic}^{\mathrm{str}}(C^\infty(M)) = \mathrm{Diffeo}(M) \ltimes \check{H}^2(M, \mathbb{Z}) = \mathsf{Pic}(C^\infty(M)). \tag{3.11}$$

Remark 3.2 In the approach of [5, 7–9] one main point was to replace the real numbers $\mathbb{R}$ by an arbitrary *ordered ring* R and $\mathbb{C}$ by the ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathrm{i}^2 = -1$. This allows to include also the formal star product algebras from deformation quantization into the game. They are defined as algebras over the formal power series $\mathbb{C}[[\lambda]]$. Surprisingly, essentially all of the constructions involving positivity go through without problems.

4 The Covariant Situation

Let us now pass to the covariant situation: we want to incorporate some given symmetry of the *-algebras in question. Here we have two main motivations and examples from differential geometry: First, a smooth action $\Phi : M \times G \longrightarrow M$ of a Lie group G on M, where by convention we choose a right action in order to have a left action $g \mapsto \Phi_g^*$ on $C^\infty(M)$ by *-automorphisms. Second, as infinitesimal version of Φ, a Lie algebra action, i.e. a Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ from a real finite dimensional Lie algebra $\mathfrak{g}$ into the Lie algebra of real vector fields, which correspond to the *-derivations of $C^\infty(M)$.

In order to formalize and unify both situations it is advantageous to consider Hopf *-algebras and their actions on algebras. Thus let H be a *Hopf *-algebra*, i.e. a unital *-algebra with a coassociative coproduct Δ, a counit ϵ and an antipode S such that $\Delta : H \longrightarrow H \otimes H$ as well as $\epsilon : H \longrightarrow \mathbb{C}$ are *-homomorphisms and $S(S(g^*)^*) = g$ for all $g \in H$, see e.g. [17, Sect. IV.8]. For the coproduct we shall use Sweedler's notation $\Delta(g) = g_{(1)} \otimes g_{(2)}$.

The two geometric examples we want to discuss are now encoded in the following Hopf *-algebras:

First, recall that any group G defines its group algebra $\mathbb{C}[G]$ which becomes a Hopf *-algebra by setting $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1} = g^*$ for $g \in G \subseteq \mathbb{C}[G]$. In this case $H = \mathbb{C}[G]$ is even *cocommutative*, i.e. $\Delta = \Delta^{\mathrm{opp}}$ where the opposite coproduct is defined by $\Delta^{\mathrm{opp}}(g) = g_{(2)} \otimes g_{(1)}$.

Second, for any real Lie algebra the complexified universal enveloping algebra $U_{\mathbb{R}}(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C} = U_{\mathbb{C}}(\mathfrak{g})$ becomes a Hopf *-algebra by setting $\Delta(\xi) = \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi$, $\epsilon(\xi) = 0$ and $S(\xi) = -\xi = \xi^*$ together with the resulting extensions to all of $U_{\mathbb{C}}(\mathfrak{g})$. Again, $U_{\mathbb{C}}(\mathfrak{g})$ is cocommutative.

The situation that a group G acts by *-automorphisms on a *-algebra as well as a Lie algebra representation by *-derivations can be unified in terms of Hopf *-algebras as follows: A **-action* $\triangleright$ of H on $\mathcal{A}$ is a bilinear map $\triangleright : H \times \mathcal{A} \longrightarrow \mathcal{A}$ such that $g \triangleright (h \triangleright a) = (gh) \triangleright a$ and $\mathbb{1}_H \triangleright a = a$, i.e. $\mathcal{A}$ is a left H-module, and $g \triangleright (ab) = (g_{(1)} \triangleright a)(g_{(2)} \triangleright b)$, $g \triangleright \mathbb{1}_{\mathcal{A}} = \epsilon(g)\mathbb{1}_{\mathcal{A}}$, and $(g \triangleright a)^* = S(g)^* \triangleright a^*$ for all $g, h \in H$ and $a, b \in \mathcal{A}$. Then it is well-known and easy to see that for our two examples $\mathbb{C}[G]$ and $U_{\mathbb{C}}(\mathfrak{g})$ this indeed generalizes and unifies the action by *-automorphisms and *-derivations, respectively. The interesting relations are all encoded in the different coproducts.

In principle and probably even more naturally, one should consider coactions instead of actions of H, see e.g. [17, Sect. III.6]. Nevertheless, we stick to the more intuitive point of view where H 'acts'.

Now suppose we have *-algebras $\mathcal{A}$, $\mathcal{B}$ with a *-action of H. Let furthermore ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a strong or *-Morita equivalence bimodule. Then we call the bimodule *H-covariant* if there is an H-module structure on $\mathcal{E}$ denoted by $\triangleright$, too, such that we have the following compatibilities

$$g \triangleright (b \cdot x) = (g_{(1)} \triangleright b) \cdot (g_{(2)} \triangleright x) \tag{4.1}$$

$$g \triangleright (x \cdot a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a) \tag{4.2}$$

$$g \triangleright {}_{\mathcal{B}}\langle x, y\rangle = {}_{\mathcal{B}}\langle g_{(1)} \triangleright x, S(g_{(2)})^* \triangleright y\rangle \tag{4.3}$$

$$g \triangleright \langle x, y\rangle_{\mathcal{A}} = \langle S(g_{(1)})^* \triangleright x, g_{(2)} \triangleright y\rangle_{\mathcal{A}} \tag{4.4}$$

for all $x, y \in \mathcal{E}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $g, h \in H$. Of course, in the case of ring-theoretic Morita theory one only requires (4.1) and (4.2). Taking isometric isomorphism classes also respecting the action of H gives the H-covariant flavours of the Picard groupoids, denoted by Pic_H, Pic^*_H, and

$\mathsf{Pic}^{\mathrm{str}}_H$, respectively. Since we can successively forget the additional structures we get the following commuting diagram of canonical groupoid morphisms:

$$
\begin{tikzcd}
\mathsf{Pic}^{\mathrm{str}}_H \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Pic}^{*}_H \arrow[dl] \arrow[dd] \\
& \mathsf{Pic}_H \arrow[dd] & \\
\mathsf{Pic}^{\mathrm{str}} \arrow[rr] \arrow[dr] & & \mathsf{Pic}^{*}, \arrow[dl] \\
& \mathsf{Pic} &
\end{tikzcd}
\tag{4.5}
$$

Now let us interpret the diagram on the level of Picard groups and in our geometric situation: Let e.g. $H = U_{\mathbb{C}}(\mathfrak{g})$ and $\mathcal{A} = C^\infty(M)$ be as before, equipped with an action of $\mathfrak{g}$ by *-derivations. Then the kernel and the image of the group morphism

$$
\mathsf{Pic}_H(\mathcal{A}) \longrightarrow \mathsf{Pic}(\mathcal{A}) \tag{4.6}
$$

encodes on which line bundles we can *lift* the $\mathfrak{g}$-action and if so, in how many different ways up to isomorphism. Analogously, in the strong situation one requires in addition compatibility with the Hermitian fiber metric. The case of $H = \mathbb{C}[G]$ leads to the question of existence and uniqueness of liftings of the group action on M to a group action on L by vector bundle automorphisms. In the strong case one requires the lift in addition to be unitary with respect to the fiber metric. All this can easily be seen from the compatibility requirements (4.1), (4.2), (4.3), and (4.4) applied to our situation.

Clearly, all these lifting problems are very natural questions in differential geometry and have been discussed by various authors, see in particular [18,21,25,28], whence one can rely on the techniques developed there. Even though our approach does not give essential new techniques to attack the (in general quite difficult) lifting problem, it shines some new light on it and unreveals some additional structure of the problem, namely the groupoid structures together with the canonical groupoid morphisms (4.5). Moreover, this point of view embeds the lifting problem in some larger and completely algebraic context since neither the *-algebras have to be commutative nor has the Hopf *-algebra to be cocommutative. As remarked already, we can even replace $\mathbb{R}$ and $\mathbb{C}$ by R and C, respectively, and incorporate in particular the formal star product algebras from deformation quantization as well.

5 The case of a Lie algebra

In this last section we consider the case of $H = U_{\mathbb{C}}(\mathfrak{g})$ more closely and develop some general results from [16] slightly further.

Assume that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a *-Morita equivalence bimodule which allows for a lift of the actions of H on $\mathcal{A}$ and $\mathcal{B}$. Then it was shown in [16, Thm 4.14] in full generality that the possible lifts are parametrized by the following group $U(H, \mathcal{A})$:

We consider linear maps $\mathsf{Hom}(H, \mathcal{A})$ with the usual convolution product given by $(\mathsf{a} * \mathsf{b})(g) = \mathsf{a}(g_{(1)})\mathsf{b}(g_{(2)})$. This makes $\mathsf{Hom}(H, \mathcal{A})$ an associative algebra with unit $\mathsf{e}(g) = \epsilon(g)\mathbb{1}_{\mathcal{A}}$, see e.g. [17, Sect. III.3]. Then we consider the following conditions for $\mathsf{a} \in \mathsf{Hom}(H, \mathcal{A})$

$$
\mathsf{a}(\mathbb{1}_H) = \mathbb{1}_{\mathcal{A}}, \tag{5.1}
$$

$$
\mathsf{a}(gh) = \mathsf{a}(g_{(1)})(g_{(2)} \triangleright \mathsf{a}(h)) \quad \text{for all} \quad g, h \in H, \tag{5.2}
$$

$$(g_{(1)} \triangleright b)\mathsf{a}(g_{(2)}) = \mathsf{a}(g_{(1)})(g_{(2)} \triangleright b) \quad \text{for all} \quad b \in \mathcal{A}, g \in H, \tag{5.3}$$

$$\mathsf{a}(g_{(1)})\mathsf{a}\left(S(g^*_{(2)})\right)^* = \epsilon(g)\mathbb{1}_{\mathcal{A}} \quad \text{for all} \quad g \in H. \tag{5.4}$$

Then $U(H, \mathcal{A})$ is defined to be the subset of those $\mathsf{a} \in \mathsf{Hom}(H, \mathcal{A})$ which satisfy (5.1)–(5.4) and it turns out that $U(H, \mathcal{A})$ is a group with respect to the convolution product [16, App. A]. Moreover, for unitary central elements $c \in U(\mathcal{Z}(\mathcal{A}))$ one defines $\hat{c} \in U(H, \mathcal{A})$ by $\hat{c}(g) = c(g \triangleright c^{-1})$. Then one obtains the exact sequence

$$1 \longrightarrow U(\mathcal{Z}(\mathcal{A}))^H \longrightarrow U(\mathcal{Z}(\mathcal{A})) \xrightarrow{\;\widehat{}\;} U(H, \mathcal{A}) \tag{5.5}$$

and the image $\widehat{U(\mathcal{Z}(\mathcal{A}))} \subseteq U(H, \mathcal{A})$ is a central and hence normal subgroup. Thus we can define the group $U_0(H, \mathcal{A}) = U(H, \mathcal{A})/\widehat{U(\mathcal{Z}(\mathcal{A}))}$.

The parametrization of all possible lifts is obtained by a free and transitive group action $\triangleright \mapsto \triangleright^{\mathsf{b}}$ of $U(H, \mathcal{B})$ on the set of lifts given by

$$g \triangleright^{\mathsf{b}} x = \mathsf{b}(g_{(1)}) \cdot (g_{(2)} \triangleright x), \tag{5.6}$$

where $\mathsf{b} \in U(H, \mathcal{B})$, $g \in H$ and $x \in \mathcal{E}$. In fact, for H-covariantly *-Morita equivalent algebras $\mathcal{A}$ and $\mathcal{B}$ we have $U(H, \mathcal{A}) \cong U(H, \mathcal{B})$. Moreover, $\triangleright^{\mathsf{b}}$ and $\triangleright$ give isomorphic actions iff $\mathsf{b} = \hat{c}$ for some $c \in U(\mathcal{Z}(\mathcal{B}))$ whence the isomorphism classes of lifts are parametrized by the group $U_0(H, \mathcal{B}) \cong U_0(H, \mathcal{A})$ which acts freely and transitively via (5.6) on the isomorphism classes of lifts.

While the above characterization works in full generality we want to specialize now to Lie algebra actions where $H = U_{\mathbb{C}}(\mathfrak{g})$. First, it follows from [16, Prop. A.7] that $U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A}) = U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{Z}(\mathcal{A}))$ and hence $U_0(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A}) = U_0(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{Z}(\mathcal{A}))$ since $U_{\mathbb{C}}(\mathfrak{g})$ is cocommutative. Thus the values of $\mathsf{a} \in U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ are automatically central, essentially by (5.3). Moreover, the groups $U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ and $U_0(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ are abelian. Since the center $\mathcal{Z}(\mathcal{A})$ is invariant under the $\mathfrak{g}$-action (by derivations!) we can restrict $\mathsf{a} \in U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ to $\mathfrak{g} \subseteq U_{\mathbb{C}}(\mathfrak{g})$ and obtain a Chevalley-Eilenberg cochain $\alpha = \mathsf{a}\big|_{\mathfrak{g}} \in C^1_{\mathrm{CE}}(\mathfrak{g}, \mathcal{Z}(\mathcal{A}))$. Evaluating the conditions (5.2), (5.3) and (5.4) on elements $\xi, \eta \in \mathfrak{g}$ we find by a simple computation the following lemma:

Lemma 5.1 *The restriction gives an injective group homomorphism*

$$U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A}) \ni \mathsf{a} \mapsto \alpha = \mathsf{a}\big|_{\mathfrak{g}} \in Z^1_{\mathrm{CE}}\left(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}}\right) \tag{5.7}$$

into the Chevalley-Eilenberg one-cocycles with values in the anti Hermitian central elements of $\mathcal{A}$.

The injectivity easily follows from successively applying (5.2).

Conversely, given $\alpha \in Z^1_{\mathrm{CE}}(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}})$ we can construct an element $\mathsf{a} \in U(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ with $\mathsf{a}\big|_{\mathfrak{g}} = \alpha$: Let $T^k_{\mathbb{C}}(\mathfrak{g})$ denote the k-th complexified tensor power of $\mathfrak{g}$ and define $\mathsf{a}^{(k)} : T^k_{\mathbb{C}}(\mathfrak{g}) \longrightarrow \mathcal{A}$ inductively by

$$\mathsf{a}^{(0)} = \mathbb{1}_{\mathcal{A}} \quad \text{and} \quad \mathsf{a}^{(k)}(\xi \otimes Y) = \alpha(\xi)\mathsf{a}^{(k-1)}(Y) + \xi \triangleright \mathsf{a}^{(k-1)}(Y) \quad \text{for} \quad k \geq 1, \tag{5.8}$$

where $Y \in T^{k-1}_{\mathbb{C}}(\mathfrak{g})$. Then a lenghty but straightforward computation using $\delta_{\mathrm{CE}}\alpha = 0$ shows that $a = \sum_{k=0}^{\infty} a^{(k)}$ passes to the universal enveloping algebra $U_{\mathbb{C}}(\mathfrak{g})$, viewed as a quotient of $T^\bullet_{\mathbb{C}}(\mathfrak{g})$ in the usual way, and fulfills (5.1) to (5.4). Thus we have:

Theorem 5.2 *The map* (5.7) *is an isomorphism of abelian groups.*

Note that this is in some sense surprising as the condition for α to be a cocycle is linear while the condition (5.2) for a is highly *non-linear*. It only becomes linear when evaluated on $\xi, \eta \in \mathfrak{g} \subseteq U_{\mathbb{C}}(\mathfrak{g})$ thanks to the fact that these elements are primitive, i.e. satisfy $\Delta(\xi) = \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi$. Thus a simplification like in Theorem 5.2 cannot be expected for more non-trivial Hopf *-algebras.

Moreover, under the identification (5.7) the elements $\hat{c}$ give just the cocycles $\hat{c}(\xi) = c(\xi \triangleright c^{-1})$ as usual. Note that in general $\widehat{U(\mathcal{Z}(\mathcal{A}))} \subseteq Z^1_{\mathrm{CE}}(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}})$ are *not* CE-coboundaries. Thus, if we want to relate $U_0(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ to Lie algebra cohomology we have to assume an additional structure for $\mathcal{A}$:

Definition 5.3 *Let $\mathcal{A}$ be a unital *-algebra. Then an exponential function* exp *is a map* $\exp : \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{Z}(\mathcal{A})$ *such that*

1. $\exp(a + b) = \exp(a)\exp(b)$,
2. $\exp(0) = \mathbb{1}_{\mathcal{A}}$,
3. $D\exp(a) = \exp(a) Da$,
4. $\exp(a^*) = \exp(a)^*$,

for all $a, b \in \mathcal{Z}(\mathcal{A})$ and $D \in \mathsf{Der}(\mathcal{A})$.

Note that $D \in \mathsf{Der}(\mathcal{A})$ induces an outer derivation $D\big|_{\mathcal{Z}(\mathcal{A})}$ of the center.

We shall now assume that $\mathcal{A}$ has an exponential function where our motivating example is of course $\mathcal{A} = C^\infty(M)$ with the usual exponential.

The first trivial observation is that for $a \in \mathcal{Z}(\mathcal{A})$ we have

$$\widehat{\exp(a)}(\xi) = -(\delta_{CE} a)(\xi), \tag{5.9}$$

and for $a = -a^* \in \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}}$ we clearly have $\exp(a) \in U(\mathcal{Z}(\mathcal{A}))$. Thus in this case $\widehat{U(\mathcal{Z}(\mathcal{A}))}$ contains all anti Hermitian CE-coboundaries in $Z^1_{\mathrm{CE}}(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}})$. Note however, that in general, $\widehat{U(\mathcal{Z}(\mathcal{A}))}$ is strictly larger. To measure this we consider those elements in $U(\mathcal{Z}(\mathcal{A}))$ which are not in the image of exp: we define the abelian group

$$\mathrm{H}^1_{\mathrm{dR}}(\mathcal{Z}(\mathcal{A}), 2\pi\mathrm{i}\mathbb{Z}) = \frac{U(\mathcal{Z}(\mathcal{A}))}{\exp\left(\mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}}\right)}, \tag{5.10}$$

where the left hand side is of course only a symbol. However, for $\mathcal{A} = C^\infty(M)$ we obtain indeed the 2πi-integral first de Rham cohomology of M by the right hand side of (5.10) which motivates our notation. Since $U(\mathcal{Z}(\mathcal{A}))$ is mapped via $c \mapsto \hat{c}$ into the cocycles $Z^1_{\mathrm{CE}}(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}})$ and $\exp(\mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}})$ gives the coboundaries via (5.9) we obtain a well-defined induced map

$$\mathrm{H}^1_{\mathrm{dR}}(\mathcal{Z}(\mathcal{A}), 2\pi\mathrm{i}\mathbb{Z}) \stackrel{\widehat{\ }}{\longrightarrow} H^1_{\mathrm{CE}}\left(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}}\right). \tag{5.11}$$

Collecting all the results we obtain the following statement:

Theorem 5.4 *Assume $\mathcal{A}$ has an exponential function. Then the restriction* (5.7) *induces a canonical group isomorphism*

$$U_0(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A}) \cong \frac{H^1_{\mathrm{CE}}\left(\mathfrak{g}, \mathcal{Z}(\mathcal{A})_{\mathrm{antiHermitian}}\right)}{\mathrm{H}^1_{\mathrm{dR}}(\widehat{\mathcal{Z}(\mathcal{A})}, 2\pi\mathrm{i}\mathbb{Z})}. \tag{5.12}$$

In particular, this applies to $\mathcal{A} = C^\infty(M)$ whence we obtain the full classification of the inequivalent lifts of the Lie algebra action to line bundles. Note that in general, $U_0(U_{\mathbb{C}}(\mathfrak{g}), \mathcal{A})$ does not depend on the line bundle itself but is universal for all line bundles.

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